SIGN CHANGES OF FOURIER COEFFICIENTS FOR HOLOMORPHIC ETA-QUOTIENTS

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Abstract. In this paper we study sign changes of an infinite class of η-quotients that are holomorphic modular forms. We also establish a connection with Hurwitz class numbers.

1. Introduction and statement of results

There is wide interest in sign changes of the coefficients of q-series $f(q) = \sum_{n\in\mathbb{Z}} c(n)q^n$ with $c(n) \in \mathbb{R}$. If $f(q)$ is the Fourier expansion of a cusp form of positive real weight on some congruence subgroup, then Knopp, Kohnen, and Pribitkin [16, Theorem 1] showed that the $c(n)$ change signs infinitely often. Sign changes along special subsets of $\mathbb{N}$ for half-integral weight cusp forms were considered in [8, 20]. Moreover, Kowalski, Lau, Soundararajan, and Wu [18, Corollary 2] proved that if f is an integral weight normalized newform, then the signs $\mathrm{sgn}(c(p))$, as p ranges over the primes, uniquely determine f. These results do not extend to general holomorphic modular forms. For example, the Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ have at most one sign change, and half of them have no sign change, so the signs of their Fourier coefficients do not uniquely determine them. Define the *Dedekind η-function*

$$\eta(z) := q^{\frac{1}{24}} \prod_{n\geq 1} (1-q^n) \qquad \left(q := e^{2\pi iz}\right).$$

In this paper, we investigate sign changes of Fourier coefficients of η-quotients $\prod_{j=1}^{m} \eta(jz)^{\delta_j}$ with $m \in \mathbb{N}$ and $\delta_j \in \mathbb{Z}$ for $1 \leq j \leq m$. Many such quotients are connected to combinatorial counting problems via the corresponding products

$$\prod_{j=1}^{m} \left(q^j;q^j\right)_\infty^{\delta_j} =: \sum_{n\geq 0} C_{1^{\delta_1}2^{\delta_2}\cdots m^{\delta_m}}(n)q^n,$$

where $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$ for $n \in \mathbb{N}_0 \cup \{\infty\}$ is the *q-Pochhammer symbol*. For example, Euler showed that $\frac{1}{\eta(z)}$ is, up to a simple factor, the partition generating function.

Numerous interesting examples involving sign changes of $C_{1^{\delta_1}2^{\delta_2}\cdots m^{\delta_m}}(n)$ appear in the literature. For example, Andrews and Lewis [2, Conjecture 2] made a conjecture pertaining to the so-called crank of a partition that is closely related to $\mathrm{sgn}(C_{1^2 3^{-1}}(n))$. This conjecture was proved by Kane [15, Corollary 2]. Motivated by work of Borwein [4], Andrews [1, Theorem 2.1] showed that $\mathrm{sgn}(C_{1^1 p^{-1}}(n))$ (p prime) is periodic with period p. Schlosser and Zhou [25] further investigated $\mathrm{sgn}(C_{1^\delta p^{-\delta}}(n))$ for various choices of δ. In all of these examples, $\mathrm{sgn}(C_{1^\delta p^{-\delta}}(n))$ exhibit a regular pattern, in contrast to the case of newforms, where the signs of the Fourier coefficients uniquely determine the form. In this paper, we are interested

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in other cases for which the signs of $C_{1^{\delta_1}2^{\delta_2}\cdots m^{\delta_m}}(n)$ satisfy a regular pattern. We focus on holomorphic η-quotients, investigating similar questions for weakly holomorphic modular forms in a parallel paper [6]. In this paper, we study several types of regular sign patterns. We first present examples for which $\operatorname{sgn}\left(C_{1^{\delta_1}2^{\delta_2}\cdots m^{\delta_m}}(n)\right)$ has some period $M \in \mathbb{N}$.

We start with $M = 1$. A number of such examples with fixed sign arise from combinatorial counting problems. For instance, Chen and Garvan [9, (1.9) and Theorem 2.1] investigated $\frac{\eta(2z)^2\eta(3z)^3}{\eta(z)^2}$, showing that for $n \in \mathbb{N}_0$ we have

$$C_{1^{-2}2^23^3}(n) = \frac{1}{24}r_3(24n+11) > 0,$$

where $r_3(n)$ is the number of representations of n as a sum of 3 squares. We obtain a similar result for $\frac{\eta(2z)^3\eta(4z)^2}{\eta(z)^2}$.

Theorem 1.1. *For $n \in \mathbb{N}_0$, we have*

$$C_{1^{-2}2^34^2}(n) > 0.$$

For $M = 2$, we have the example $\frac{\eta(z)^3\eta(3z)^3}{\eta(2z)^2}$.

Theorem 1.2. *The sequence $\{\operatorname{sgn}(C_{1^32^{-2}3^3}(n))\}_{n\geq 1}$ has period 2. In particular, we have*

$$(-1)^n C_{1^32^{-2}3^3}(n) > 0.$$

For $M = 3$, we find the example $\frac{\eta(z)^4\eta(2z)^4}{\eta(3z)^2}$.

Theorem 1.3. *The sequence $\{\operatorname{sgn}(C_{1^42^43^{-2}}(n))\}_{n\geq 1}$ has period 3. In particular, we have*

$$\operatorname{sgn}\left(C_{1^42^43^{-2}}(n)\right) = \begin{cases} 1 & \text{if } 3 \mid n, \\ -1 & \text{otherwise.} \end{cases}$$

For $M = 8$, we find two examples, $\frac{\eta(z)^4\eta(2z)^2}{\eta(4z)^2}$ and $\frac{\eta(z)^4\eta(2z)^4}{\eta(4z)^3}$.

Theorem 1.4.

(1) *The sequence $\{\operatorname{sgn}(C_{1^42^24^{-2}}(n))\}_{n\geq 1}$ has period 8. In particular, we have*

$$\operatorname{sgn}\left(C_{1^42^24^{-2}}(n)\right) = \begin{cases} 1 & \text{if } n \equiv 0, 3, 7 \pmod 8, \\ -1 & \text{if } n \equiv 1, 4, 5 \pmod 8, \\ 0 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

(2) *The sequence $\{\operatorname{sgn}(C_{1^42^44^{-3}}(n))\}_{n\geq 1}$ has period 8. In particular, we have*

$$\operatorname{sgn}\left(C_{1^42^44^{-3}}(n)\right) = \begin{cases} 1 & \text{if } n \equiv 0, 3, 6, 7 \pmod 8, \\ -1 & \text{if } n \equiv 1, 2, 4, 5 \pmod 8. \end{cases}$$

For a prime p, we next consider sign changes for the pair of infinite families of η-quotients

$$Q_p(z) := \frac{\eta(z)^{p^2}}{\eta(pz)^p}, \quad P_p(z) := \frac{\eta(z)^p}{\eta(pz)}.$$

These families have appeared throughout the literature, and explicit Fourier expansions are known for certain small primes p. The function Q_2 is an Eisenstein series (see [17, Example

10.6 (10.14)]) and P_2 is a theta function (see [23, Theorem 1.60])

$$Q_2(z) = 1 - 4\sum_{n\geq 1}(-1)^{n+1}\sum_{d|n}\left(\frac{-1}{d}\right)q^n, \qquad P_2(z) = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2},$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the extended Legendre symbol. The Fourier expansion (see [17, Example 11.4])

$$P_3(z) = 1 - 3\sum_{n\geq 1}\sum_{d|n}\left(\frac{d}{3}\right)q^n + 9\sum_{n\geq 1}\sum_{d|n}\left(\frac{d}{3}\right)q^{3n}$$

closely resembles the Fourier expansion of Q_2. These functions are *lacunary* (see [11, Theorem 1.2] for the statement for Q_2), meaning that the set of n for which the n-th Fourier coefficient is non-zero has density zero, and hence their Fourier coefficients cannot exhibit regular sign changes. Although the Fourier expansion (see [17, Example 12.16 (12.34)])

$$P_5(z) = 1 - 5\sum_{n\geq 1}\sum_{d|n}\left(\frac{d}{5}\right)dq^n$$

also resembles the Fourier expansions of Q_2 and P_3, we next see that the signs of the Fourier coefficients of P_5 satisfy a somewhat regular pattern related to the prime factorization of n which is part of a more general phenomenon for larger p.

Theorem 1.5.

(1) *Let $p \geq 3$ be prime and $a \in \mathbb{N}_0$. Then, for sufficiently large m coprime to p, we have*

$$\operatorname{sgn}\left(C_{1^{p^2}p^{-p}}\left(p^a m\right)\right) = \left(\frac{2}{p}\right)\left(\frac{m}{p}\right).$$

(2) *Let $p \geq 5$ be prime and $a \in \mathbb{N}_0$. Then, for m sufficiently large coprime to p, we have*

$$\operatorname{sgn}\left(C_{1^p p^{-1}}\left(p^a m\right)\right) = \left(\frac{-2}{p}\right)\left(\frac{m}{p}\right).$$

Although the sign changes asserted in Theorem 1.5 only hold for m sufficiently large, in special cases the formula holds for all n if the underlying space of cusp forms is trivial.

Corollary 1.6.

(1) *For $a \in \mathbb{N}_0$ and $m \in \mathbb{N}$ with $\gcd(3, m) = 1$, we have*

$$\operatorname{sgn}\left(C_{1^9 3^{-3}}\left(3^a m\right)\right) = -\left(\frac{m}{3}\right).$$

(2) *For $a \in \mathbb{N}_0$ and $m \in \mathbb{N}$ with $\gcd(5, m) = 1$, we have*

$$\operatorname{sgn}\left(C_{1^5 5^{-1}}\left(5^a m\right)\right) = -\left(\frac{m}{5}\right).$$

Finally, we consider a half-integral weight case for which the sign of the n-th Fourier coefficient resembles Corollary 1.6.

Theorem 1.7. *For $n \in \mathbb{N}_0$, write $8n + 1 = 3^a m$ with $3 \nmid m$. Then we have*

$$\operatorname{sgn}\left(C_{1^2 2^2 3^{-1}}(n)\right) = \begin{cases} \left(\frac{m}{3}\right) & \text{if } a = 0, \\ -\frac{\left(\frac{m}{3}\right)+1}{2} & \text{if } a = 1, \\ -1 & \text{if } a = 2, \\ \operatorname{sgn}\left(C_{1^2 2^2 3^{-1}}\left(\frac{n-1}{9}\right)\right) & \text{if } a \geq 3. \end{cases}$$

The paper is organized as follows. In Section 2, we recall some basic facts on modular forms and Hurwitz class numbers. In Section 3, we prove Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.2. In Section 5 we show Theorem 1.3, in Section 6 we prove Theorem 1.4, and in Section 7 we show Theorem 1.5, Corollary 1.6, and Theorem 1.7.

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2. Preliminaries

2.1. **Modular forms.** We briefly recall the relevant definitions and some basic facts about modular forms and refer the reader to [23] for details. As usual, for d odd, we let

$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

For $k \in \frac{1}{2}\mathbb{Z}$, $N \in \mathbb{N}$ ($4 \mid N$ if $k \in \mathbb{Z} + \frac{1}{2}$), and a character $\chi \pmod N$, a function $f : \mathbb{H} \to \mathbb{C}$ satisfies *modularity of weight k on $\Gamma_0(N)$ with character χ* if for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$

$$f|_k\gamma = \chi(d)f.$$

Here the weight k *slash operator* is defined by

$$f\big|_k\gamma(z) := \begin{cases} \left(\frac{c}{d}\right)\varepsilon_d^{2k}(cz+d)^{-k}f(\gamma z) & \text{if } k \in \mathbb{Z} + \frac{1}{2}, \\ (cz+d)^{-k}f(\gamma z) & \text{if } k \in \mathbb{Z}. \end{cases}$$

We call $f : \mathbb{H} \to \mathbb{C}$ a *holomorphic modular form of weight k on $\Gamma_0(N)$ with character χ* if f is holomorphic on $\mathbb{H}$, satisfies modularity of weight k on $\Gamma_0(N)$ with character χ, and for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, $(cz+d)^{-k}f(\gamma z)$ is bounded as $z \to i\infty$. We denote the space of such forms by $M_k(\Gamma_0(N), \chi)$. Modular forms for which $(cz+d)^{-k}f(\gamma z)$ vanishes as $z \to i\infty$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ are called *cusp forms*. The corresponding subspace is denoted by $S_k(\Gamma_0(N), \chi)$. We drop χ from the notation if it is trivial. The *Petersson inner product* between $f, g \in S_k(\Gamma_0(N), \chi)$ is defined by ($z = x + iy$)

$$\langle f, g\rangle := \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \int_{\Gamma_0(N)\backslash\mathbb{H}} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2}.$$

In order to show identities between modular forms, we use the following consequence of the valence formula.

Lemma 2.1. *Let $k \in \frac{1}{2}\mathbb{N}$, $N \in \mathbb{N}$, and χ be a character* (mod N). *If $f \in M_k(\Gamma_0(N), \chi)$ satisfies $c_f(n) = 0$ for every $0 \le n \le N\frac{k}{12}\prod_{p\mid N}(1 + \frac{1}{p})$, then $f = 0$.*

2.2. **Operators on (non-holomorphic) modular forms.** We recall the action of certain operators on non-holomorphic functions satisfying modularity of weight $\kappa \in \frac{1}{2}\mathbb{Z}$ on some group $\Gamma_0(N)$ with some character. For $f(z) = \sum_{n\in\mathbb{Z}} c_{f,y}(n)q^n$ (we omit the dependence on y if f is holomorphic), we let

$$f \mid U_\ell(z) := \sum_{n\in\mathbb{Z}} c_{f,\frac{y}{\ell}}(\ell n)q^n, \quad f \mid V_\ell(z) := f(\ell z).$$

For $M \in \mathbb{N}$ and $m \in \mathbb{N}_0$, we define the *sieving operator*

$$f \mid S_{M,m}(z) := \sum_{\substack{n\in\mathbb{Z} \\ n\equiv m \,(\mathrm{mod}\ M)}} c_{f,y}(n)q^n.$$

Moreover, for a character ψ, we define the *twist of f by ψ* by

$$f \otimes \psi(z) := \sum_{n\in\mathbb{Z}} \psi(n)c_{f,y}(n)q^n.$$

To state the modular properties of these functions, we require the *radical* $\mathrm{rad}(n) := \prod_{p\mid n} p$ and the character $\chi_D(n) := (\frac{D}{n})$. Recall that the *conductor* of a character $\chi \pmod N$ is the smallest $M \in \mathbb{N}$ such that for all $n \in \mathbb{Z}$ with $\gcd(n, N) = 1$, we have $\chi(n + M) = \chi(n)$. The following properties are well-known; see, for example, [7, Lemma 2.3] for a proof.

Lemma 2.2. *Suppose that $f : \mathbb{H} \to \mathbb{C}$ satisfies modularity of weight $k \in \mathbb{Z} + \frac{1}{2}$ on $\Gamma_0(N)$ $(4 \mid N)$ with character χ of conductor $N_\chi \mid N$.*

(1) *For $\delta \in \mathbb{N}$, $f|U_\delta$ satisfies modularity of weight k on $\Gamma_0(4\,\mathrm{lcm}(\frac{N}{4}, \mathrm{rad}(\delta)))$ with character $\chi\chi_{4\delta}$.*
(2) *Suppose that $M \mid 24$ and $M \not\equiv 2 \pmod 4$. Then $f|S_{M,m}$ satisfies modularity of weight k on $\Gamma_0(\mathrm{lcm}(N, M^2, MN_\chi))$ with character χ.*
(3) *For $\delta \in \mathbb{N}$, $f|V_\delta$ satisfies modularity of weight k on $\Gamma_0(N\delta)$ with character $\chi\chi_{4\delta}$.*

The following lemma may be shown similarly to Theorem 2.2.

Lemma 2.3. *Let $N \in \mathbb{N}$, χ be a character $\pmod N$ with conductor $N_\chi \mid N$, and $k \in \mathbb{N}$. Suppose that f satisfies modularity of weight k on $\Gamma_0(N)$ with character χ.*

(1) *For $\delta \in \mathbb{N}$, $f|V_\delta$ satisfies modularity of weight k on $\Gamma_0(N\delta)$ with character χ.*
(2) *If $M \in \mathbb{N}$ with $M \mid 24$ and $m \in \mathbb{Z}$, then $f|S_{M,m}$ satisfies modularity of weight k on $\Gamma_0(\mathrm{lcm}(N, M^2, MN_\chi))$ with character χ.*
(3) *For $M \in \mathbb{N}$ and a character $\psi \pmod M$, the twist $f \otimes \psi$ satisfies modularity of weight k for $\Gamma_0(\mathrm{lcm}(N, M^2, MN_\chi))$ with character $\chi\psi^2$.*
(4) *If $\delta \mid N$, then $f|U_\delta$ satisfies modularity of weight k on $\Gamma_0(N)$ with character χ.*

2.3. **Eisenstein series.** For $k \in \mathbb{N}$ with $k \geq 2$ and primitive characters χ, ψ, we define the *Eisenstein series*

$$E_{k,\chi,\psi}(z) := \delta_{\chi=\chi_1} L(1-k, \psi) + 2\sum_{n\geq 1}\sum_{d\mid n} \chi\left(\frac{n}{d}\right)\psi(d)d^{k-1}q^n, \tag{2.1}$$

where the *L-function* for the character ψ is defined for $\mathrm{Re}(s) > 1$ by $L(s, \psi) := \sum_{n\geq 1} \frac{\psi(n)}{n^s}$. This function admits a meromorphic continuation to $s \in \mathbb{C}$. The modular properties of these Eisenstein series may be found in [13, Theorem 4.5.2] and [13, Theorem 4.6.2].

Lemma 2.4. *Suppose that χ and ψ are primitive characters of conductors N_χ and N_ψ, respectively. If $k > 2$ or ($k = 2$ and either χ or ψ is non-trivial), then $E_{k,\chi,\psi} \in M_k\left(\Gamma_0\left(N_\chi N_\psi\right), \chi\psi\right)$.*

For $k, N \in \mathbb{N}$ and a character $\varrho \pmod N$, we define the *Eisenstein series subspace* of $M_k(\Gamma_0(N), \varrho)$ to be the space spanned by $E_{k,\chi,\psi}|V_d$, where χ and ψ are characters satisfying $\chi\psi = \varrho$ and $d \in \mathbb{N}$ satisfies $N_\chi N_\psi d \mid N$. Here $\chi\psi = \varrho$ means that they agree as characters $\pmod N$. Every $f \in M_k(\Gamma_0(N), \varrho)$ can be written uniquely as $f = E + g$ with E contained in the Eisenstein series subspace and g a cusp form. We call E the *Eisenstein series part* of f and g the *cuspidal part* of f. If $\chi = \chi_1$ and $\psi = \chi_{\left(\frac{-1}{p}\right)p}$ for an odd prime p, then we normalize the Eisenstein series by multiplying by

$$L_{k,p} := \frac{1}{L\left(1-k, \chi_{\left(\frac{-1}{p}\right)p}\right)}.$$

Using the Euler–Maclaurin summation formula (see [28, (44)]), one can determine the behavior of the Eisenstein series at 0.

Lemma 2.5. *Suppose that $k \geq 2$ and p is an odd prime. Then $L_{k,p}E_{k,\chi_1,\chi_{\left(\frac{-1}{p}\right)p}}$ has constant term 1 at $i\infty$ and vanishes at 0.*

In addition to the holomorphic Eisenstein series, we also require the quasimodular Eisenstein series of weight 2, which, for $\sigma(n) := \sum_{d|n} d$, is defined by

$$E_2(z) := 1 - 24\sum_{n\geq 1}\sigma(n)q^n.$$

The Eisenstein series E_2 is not modular, but it has a natural "modular completion"

$$\widehat{E}_2(z) := E_2(z) - \frac{3}{\pi y}.$$

Although $\widehat{E}_2$ is not holomorphic, it satisfies, for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,

$$\widehat{E}_2|_2\gamma = \widehat{E}_2.$$

2.4. **Hecke operators.** In this subsection, we restrict to integral weight modular forms. For $N, k \in \mathbb{N}$, χ a character $\pmod N$, and p a prime, we define the *Hecke operator* T_p acting on $f \in M_k(\Gamma_0(N), \chi)$ by (see [23, Definition 2.1])

$$f\big|T_p(z) := \sum_{n\geq 0}\left(c_f(pn) + \chi(p)p^{k-1}c_f\left(\frac{n}{p}\right)\right)q^n,$$

where $c_f(\alpha) := 0$ for $\alpha \in \mathbb{Q}^+ \setminus \mathbb{N}_0$. There exists a natural basis of cusp forms which are simultaneous eigenfunctions under all Hecke operators T_p with $p \nmid N$. We call these simultaneous eigenfunctions *Hecke eigenforms*. If $f \in M_k(\Gamma_0(N), \chi)$ is a Hecke eigenform, then $f|V_d \in M_k(\Gamma_0(Nd), \chi)$ is also a Hecke eigenform. For $M \in \mathbb{N}$, the subspace of $M_k(\Gamma_0(M), \chi)$ spanned by $f|V_d$ with $f \in M_k(\Gamma_0(N), \chi)$ for $1 \leq N < M$ is the *old space*; the orthogonal complement of the old space is the *new space*. Hecke eigenforms in the new spaces are called *newforms*. We normalize the Fourier expansions so that $c_f(1) = 1$. Letting $d(n)$ denote the number of divisors of n, a result of Deligne [12] gives an explicit bound on $c_f(n)$.

Theorem 2.6. *(Deligne) Suppose that k, $N \in \mathbb{N}$, χ is a character* $(\mathrm{mod}\ N)$*, and $f \in S_k(\Gamma_0(N), \chi)$ is a normalized newform. Then*

$$|c_f(n)| \leq d(n) n^{\frac{k-1}{2}}.$$

Letting $\|f\| := \sqrt{\langle f, f \rangle}$ denote the Petersson norm, Schulze-Pillot and Yenirce [26, Theorem 12] constructed an explicit orthonormal basis for $S_k(\Gamma_0(N), \chi)$ and used Theorem 2.6 to obtain a bound for the Fourier coefficients of any $f \in S_k(\Gamma_0(N), \chi)$.

Lemma 2.7. *Suppose that $k, N \in \mathbb{N}$, χ is a character* $(\mathrm{mod}\ N)$*, and $f \in S_k(\Gamma_0(N), \chi)$. Then we have*

$$|c_f(n)| \leq 2\sqrt{\pi} e^{2\pi} \sqrt{\dim_{\mathbb{C}}(S_k(\Gamma_0(N), \chi))} \sqrt{N} \prod_{p|N} \frac{\left(1 + \frac{1}{p}\right)^3}{\sqrt{1 - \frac{1}{p^4}}} \|f\| d(n) n^{\frac{k-1}{2}}.$$

2.5. **L-functions of Dirichlet characters.** We require a lemma for evaluating L-functions of Dirichlet characters at non-positive integers. The following identity is well-known and follows directly from [3, p. 249] and [3, Theorem 12.13].

Lemma 2.8. *For a discriminant D and $k \in \mathbb{N}$, we have*

$$L(1 - k, \chi_D) = -\frac{|D|^{k-1}}{k} \sum_{r=1}^{|D|} \chi_D(r) B_k\left(\frac{r}{|D|}\right),$$

where $B_k(x)$ denotes the k-th Bernoulli polynomial.

For the specific case $D = (\frac{-1}{p})p$ and $k \equiv k_p \pmod 2$ with $k_p := \frac{p-1}{2}$, one may use the functional equation for the Dirichlet L-function and Gauss's evaluation of the quadratic Gauss sum to obtain the sign of the L-value in Theorem 2.8.

Lemma 2.9. *Let p be an odd prime and $k \geq 2$ be an integer satisfying $k \equiv k_p \pmod 2$. Then*

$$\mathrm{sgn}\left(L_{k,p}\right) = (-1)^{\frac{k}{2} + \frac{p-1}{4}} \left(\frac{-2}{p}\right).$$

In particular, we have

$$\mathrm{sgn}\left(L_{k_p,p}\right) = \left(\frac{-2}{p}\right) \quad \text{and} \quad \mathrm{sgn}\left(L_{pk_p,p}\right) = \left(\frac{2}{p}\right).$$

2.6. **Hurwitz class numbers.** For $D \in \mathbb{N}$, let $H(D)$ denote the *Hurwitz class number*, the weighted number of classes of positive-definite integral binary quadratic forms of discriminant $-D$, where each class is weighted by the inverse of the size of its automorphism group in $\mathrm{PSL}_2(\mathbb{Z})$. We note that if $-D < 0$ is a discriminant, then $H(D) > 0$, while $H(D) = 0$ if $-D$ is not a discriminant. We extend the definition by setting $H(0) := -\frac{1}{12}$ and $H(r) := 0$ for $r \in \mathbb{Q} \setminus \mathbb{N}_0$. For a fundamental discriminant $-D$, we have [10, p. 273]

$$H\left(Df^2\right) = H(D) S_D(f). \tag{2.2}$$

Here, letting μ denote the Möbius function, we define

$$S_D(f) := \sum_{d|f} \mu(d) \chi_{-D}(d) \sigma\left(\frac{f}{d}\right). \tag{2.3}$$

For $\ell_1, \ell_2 \in \mathbb{N}$, let

$$\mathcal{H}(z) := \sum_{D \geq 0} H(D) q^D, \quad \mathcal{H}_{\ell_1,\ell_2} := \mathcal{H} \mid (U_{\ell_1\ell_2} - \ell_2 U_{\ell_1} \circ V_{\ell_2}). \tag{2.4}$$

For $y > 0$, let $\Gamma(s, y) := \int_y^\infty e^{-t} t^{s-1} dt$ be the *incomplete gamma function*. The modularity of

$$\widehat{\mathcal{H}}(z) := \mathcal{H}(z) + \frac{1}{8\pi\sqrt{y}} + \frac{1}{4\sqrt{\pi}} \sum_{n \geq 1} n\Gamma\left(-\frac{1}{2}, 4\pi n^2 y\right) q^{-n^2}$$

was proved by Zagier [27] (see also [14, Chapter 2, Theorem 2]). Using his result, the modularity of $\mathcal{H}_{\ell_1,\ell_2}$ was given in [7, Lemma 2.6].

Lemma 2.10. *For $\ell_1, \ell_2 \in \mathbb{N}$, with $\gcd(\ell_1, \ell_2) = 1$ and ℓ_2 squarefree, we have that*

$$\mathcal{H}_{\ell_1,\ell_2} \in M_{\frac{3}{2}}\left(\Gamma_0(4\operatorname{rad}(\ell_1)\ell_2), \chi_{4\ell_1\ell_2}\right).$$

3. Proof of Theorem 1.1

We abbreviate $b_1(n) := C_{1^{-2}2^3 4^2}(n)$.

Proof of Theorem 1.1. By [23, Theorem 1.60], we have

$$\frac{\eta(16z)^2}{\eta(8z)} = \sum_{n \geq 0} q^{(2n+1)^2}.$$

We therefore have[1]

$$\sum_{n \geq 0} b_1(n) q^{8n+4} = \sum_{\boldsymbol{n} \in \mathbb{N}_0^3} q^{8\left(T_{n_1} + T_{n_2} + 2T_{n_3}\right)+4},$$

where $T_n := \frac{n(n+1)}{2}$. Comparing the $(8n+4)$-th Fourier coefficient on both sides, we see that

$$b_1(n) = \#\left\{\boldsymbol{n} \in \mathbb{N}_0^3 : T_{n_1} + T_{n_2} + 2T_{n_3} = n\right\}. \tag{3.1}$$

It was proved by Liouville [22] (see also the statement in [5] and its generalization in [5, Theorem 1.1]) that, for $n \in \mathbb{N}_0$,

$$\#\left\{\boldsymbol{n} \in \mathbb{N}_0^3 : T_{n_1} + T_{n_2} + 2T_{n_3} = n\right\} > 0.$$

Combining this with (3.1) immediately yields Theorem 1.1. □

4. Proof of Theorem 1.2

Abbreviating $b_2(n) := C_{1^3 2^{-2} 3^3}(n)$, $s_2(n) := \operatorname{sgn}(b_2(n))$, we first show the following.

Lemma 4.1. *We have*

$$b_2(n) = \begin{cases} \sum_{d \mid (3n+1)} \left(\frac{3}{d}\right) d & \text{if } n \equiv 0 \pmod 4, \\ -\frac{1}{3} \sum_{d \mid (3n+1)} \left(\frac{3}{d}\right) d & \text{if } n \equiv 2 \pmod 4, \\ -\frac{1}{3} \sum_{d \mid (3n+1)} \left(\frac{12}{d}\right) d - \frac{2}{3} \sum_{d \mid (3n+1)} \left(\frac{12}{\frac{3n+1}{d}}\right) d & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

[1] Throughout we use boldface letters for vectors.

Proof. By [23, Theorem 1.64], $\frac{\eta(3z)^3\eta(9z)^3}{\eta(6z)^2} \in M_2(\Gamma_0(72), \chi_{12})$.

The generating function of the right-hand side of Lemma 4.1 is

$$\frac{1}{2}E_{2,\chi_1,\chi_{12}}\big|S_{12,1} - \frac{1}{6}E_{2,\chi_1,\chi_{12}}\big|S_{12,7} - \frac{1}{6}E_{2,\chi_1,\chi_{12}}\big|S_{6,4} - \frac{1}{3}E_{2,\chi_{12},\chi_1}\big|S_{6,4}.$$

By Lemma 2.4, $E_{2,\chi_1,\chi_{12}}, E_{2,\chi_{12},\chi_1} \in M_2\left(\Gamma_0(12), \chi_{12}\right)$. Hence, for $m \in \mathbb{Z}$ and $M \in \mathbb{N}$ with $M \mid 12$, Lemma 2.3 (2) implies that $E_{2,\chi_1,\chi_{12}}\big|S_{M,m}, E_{2,\chi_{12},\chi_1}\big|S_{M,m} \in M_2\left(\Gamma_0(144), \chi_{12}\right)$. We conclude that the generating functions of both sides of the claimed identity lie in $M_2(\Gamma_0(144), \chi_{12})$. By Theorem 2.1, the identity holds if it holds for the first 48 Fourier coefficients. It was verified computationally for all $3n + 1 \leq 301$. □

Proof of Theorem 1.2. Writing $3n + 1 = 2^\ell m$ with $\gcd(6, m) = 1$, we have

$$\sum_{d\mid(3n+1)}\left(\frac{12}{d}\right)d = \sum_{d\mid m}\left(\frac{3}{d}\right)d\sum_{r=0}^{\ell}\left(\frac{12}{2^r}\right)2^r = \prod_{p\mid m}\sum_{j=0}^{\mathrm{ord}_p(m)}\left(\frac{3}{p^j}\right)p^j = \prod_{p\mid m}\frac{1-\left(\left(\frac{3}{p}\right)p\right)^{\mathrm{ord}_p(m)+1}}{1-\left(\frac{3}{p}\right)p}.$$

Now

$$\frac{1-\left(\left(\frac{3}{p}\right)p\right)^{\mathrm{ord}_p(m)+1}}{1-\left(\frac{3}{p}\right)p} = \begin{cases} \frac{1-p^{\mathrm{ord}_p(m)}}{1-p} & \text{if } p \equiv \pm 1 \,(\mathrm{mod}\ 12), \\ \frac{1-p^{\mathrm{ord}_p(m)}}{1+p} & \text{if } p \equiv \pm 5 \,(\mathrm{mod}\ 12),\ \mathrm{ord}_p(m) \text{ odd}, \\ \frac{1+p^{\mathrm{ord}_p(m)}}{1+p} & \text{if } p \equiv \pm 5 \,(\mathrm{mod}\ 12),\ \mathrm{ord}_p(m) \text{ even}. \end{cases}$$

This implies that

$$\mathrm{sgn}\left(\sum_{d\mid m}\left(\frac{3}{d}\right)d\right) = \left(\frac{3}{m}\right) = (-1)^{\frac{m-1}{2}}\left(\frac{m}{3}\right). \tag{4.1}$$

For n even, we have $3n + 1 = m$, so (4.1) implies that

$$\mathrm{sgn}\left(\sum_{d\mid(3n+1)}\left(\frac{3}{d}\right)d\right) = (-1)^{\frac{m-1}{2}}\left(\frac{m}{3}\right). \tag{4.2}$$

If $n \equiv 0 \,(\mathrm{mod}\ 4)$, then $m = 3n + 1 \equiv 1 \,(\mathrm{mod}\ 12)$, so plugging (4.2) into Lemma 4.1 yields that $s_2(n) = 1$. If $n \equiv 2 \,(\mathrm{mod}\ 4)$, then $m = 3n + 1 \equiv 7 \,(\mathrm{mod}\ 12)$, so inserting (4.2) into Lemma 4.1 implies that $s_2(n) = 1$. For $2 \nmid n$, we write $3n + 1 = 2^\ell m$ with m odd. Noting that $(\frac{12}{\frac{3n+1}{d}}) = 0$ unless $2^\ell \mid d$, Lemma 4.1 gives that

$$b_2(n) = -\frac{1}{3}\left(1 + 2^{\ell+1}\left(\frac{3}{m}\right)\right)\sum_{d\mid m}\left(\frac{3}{d}\right)d.$$

By (4.1), we have $\mathrm{sgn}(\sum_{d\mid m}\left(\frac{3}{d}\right)d) = (\frac{3}{m})$. Since $2^{\ell+1} > 1$, $\mathrm{sgn}(1 + 2^{\ell+1}(\frac{3}{m})) = \mathrm{sgn}((\frac{3}{m}))$. So overall we obtain $s_2(n) = -1$ for n odd. Combining these cases proves the claim. □

5. Proof of Theorem 1.3

5.1. **Decomposition of the eta-quotient.** We now explicitly decompose $\frac{\eta(4z)^4\eta(8z)^4}{\eta(12z)^2}$ into an Eisenstein series and a cusp form. For this, we define the Eisenstein series

$$\begin{aligned}\mathcal{E}(z) := &\sum_{\substack{n\geq 1\\ n\equiv 1 \,(\mathrm{mod}\ 12)}} \sum_{d|n} \left(\frac{-1}{d}\right) d^2 q^n - \frac{1}{2} \sum_{\substack{n\geq 1\\ n\equiv 5 \,(\mathrm{mod}\ 12)}} \sum_{d|n} \left(\frac{-1}{d}\right) d^2 q^n \\ &- 2 \sum_{\substack{n\geq 1\\ n\equiv 9 \,(\mathrm{mod}\ 12)}} \left(\sum_{d|n} \left(\frac{-1}{d}\right) d^2 + 6 \sum_{d|\frac{n}{3}} \left(\frac{-1}{d}\right) d^2\right) q^n \\ &= \frac{1}{2} E_{3,\chi_1,\chi_{-4}}|S_{12,1} - \frac{1}{4} E_{3,\chi_1,\chi_{-4}}|S_{12,5} - E_{3,\chi_1,\chi_{-4}}|S_{12,9} - 6E_{3,\chi_1,\chi_{-4}}|V_3|S_{4,1}.\end{aligned}$$

For the cuspidal part, let g_1 denote the normalized newform in $S_3(\Gamma_0(12),\chi_{-4})$ with

$$g_1(z) = q - \left(1+\sqrt{3}i\right) q^2 + \sqrt{3}iq^3 - 2\left(1-\sqrt{3}i\right) q^4 - 2q^5 + \left(3-\sqrt{3}i\right) q^6 - 4\sqrt{3}iq^7 + 8q^8 - 3q^9 + O\left(q^{10}\right)$$

and let g_2 be the normalized newform in $S_3(\Gamma_0(36),\chi_{-4})$ satisfying

$$g_2(z) = q - 2q^2 + 4q^4 + 8q^5 - 8q^8 - 16q^{10} - 10q^{13} + 16q^{16} - 16q^{17} + O\left(q^{20}\right).$$

We then obtain the following decomposition of the eta-quotient.

Lemma 5.1. *We have*

$$\begin{aligned}\frac{\eta(4z)^4\eta(8z)^4}{\eta(12z)^2} = \frac{1}{7}\mathcal{E}(z) - \frac{27}{14} g_1|S_{12,9}(z) + \frac{3}{14} g_1|S_{4,1}(z) + \frac{9}{14} g_1\otimes\chi_{-3}|S_{4,1}(z) \\ - \frac{3}{16} g_2|S_{4,1}(z) + \frac{3}{16} g_2\otimes\chi_{-3}|S_{4,1}(z).\end{aligned}$$

Proof. By [23, Theorem 1.64], $\frac{\eta(4z)^4\eta(8z)^4}{\eta(12z)^2} \in M_3(\Gamma_0(72),\chi_{-4})$. By Theorem 2.4, $E_{3,\chi_1,\chi_{-4}} \in M_3(\Gamma_0(4),\chi_{-4})$. Using Lemma 2.3 (2), $E_{3,\chi_1,\chi_{-4}}|S_{12,m} \in M_3(\Gamma_0(144),\chi_{-4})$ for $m\in\mathbb{Z}$. Lemma 2.3 (1) implies that $E_{3,\chi_1,\chi_{-4}}|V_3 \in M_3(\Gamma_0(12),\chi_{-4})$. Lemma 2.3 (2) then gives that $E_{3,\chi_1,\chi_{-4}}|V_3|S_{4,1} \in M_3(\Gamma_0(48),\chi_{-4})$. Thus $\mathcal{E}\in M_3\left(\Gamma_0(144),\chi_{-4}\right)$. By Lemma 2.3 (2), (3), $-\frac{27}{14}g_1|S_{12,9} + \frac{3}{14}g_1|S_{4,1} + \frac{9}{14}g_1\otimes\chi_{-3}|S_{4,1} - \frac{3}{16}g_2|S_{4,1} + \frac{3}{16}g_2\otimes\chi_{-3}|S_{4,1} \in M_3\left(\Gamma_0(144),\chi_{-4}\right)$. Thus both sides of the claimed identity lie in $M_3(\Gamma_0(144),\chi_{-4})$. To prove the identity, it suffices to check the first 72 Fourier coefficients. This was verified numerically. □

5.2. **Fourier coefficients of the Eisenstein series part.** We now determine the sign of the n-th Fourier coefficient $A(n)$ of the Eisenstein series part $\mathcal{E}$.

Lemma 5.2. *Suppose that $n\equiv 1 \,(\mathrm{mod}\ 4)$. Then we have*

$$\sum_{d|n} \left(\frac{-1}{d}\right) d^2 + 6\sum_{d|\frac{n}{3}} \left(\frac{-1}{d}\right) d^2 > 0.$$

In particular,

$$\mathrm{sgn}(A(4n+1)) = \begin{cases} 1 & \text{if } 3\mid n,\\ -1 & \text{otherwise.}\end{cases}$$

Proof. If $3 \nmid n$, then the second sum on the left-hand side of the first claim vanishes. Since $n \equiv 1 \pmod 4$, we have that n (and thus d) is odd. Thus the above becomes

$$\prod_{p|n} \sum_{j=0}^{\operatorname{ord}_p(n)} \left(\frac{-1}{p^j}\right) p^{2j} = \prod_{p|n} \begin{cases} \frac{1-p^{2\operatorname{ord}_p(n)+2}}{1-p^2} & \text{if } p \equiv 1 \pmod 4, \\ \frac{1-p^{2\operatorname{ord}_p(n)+2}}{1+p^2} & \text{if } p \equiv 3 \pmod 4,\ 2 \nmid \operatorname{ord}_p(n), \\ \frac{1+p^{2\operatorname{ord}_p(n)+2}}{1+p^2} & \text{if } p \equiv 3 \pmod 4,\ 2 \mid \operatorname{ord}_p(n). \end{cases}$$

Now only the second case yields a negative sign, so

$$\operatorname{sgn}\left(\sum_{d|n} \left(\frac{-1}{d}\right) d^2\right) = (-1)^{\#\{p|n\,:\,p\equiv 3 \pmod 4,\, 2\nmid \operatorname{ord}_p(n)\}} \equiv n \pmod 4 \tag{5.1}$$

since n is odd. Since $n \equiv 1 \pmod 4$, we conclude that the sign in this case is 1.

Finally, suppose that $3 \mid n$. Write $n = 3^\ell m$ ($\ell \in \mathbb{N}$), $3 \nmid m$. Then we need to rewrite the left-hand side of the first claim of the lemma as

$$\begin{aligned} &\sum_{d|m} \left(\frac{-1}{d}\right) d^2 \sum_{j=0}^{\ell} \left(\frac{-1}{3^j}\right) 3^{2j} + 6 \sum_{d|m} \left(\frac{-1}{d}\right) d^2 \sum_{j=0}^{\ell-1} \left(\frac{-1}{3^j}\right) 3^{2j} \\ &\qquad = \left(7 \sum_{j=0}^{\ell-1} (-1)^j 3^{2j} + (-1)^\ell 3^{2\ell}\right) \sum_{d|m} \left(\frac{-1}{d}\right) d^2. \end{aligned} \tag{5.2}$$

The sign of the sum over d is evaluated in (5.1). The expression in parentheses equals $\frac{7+(-1)^\ell 3^{2\ell+1}}{10}$. Note that the sign of this factor is $(-1)^\ell$ (because $\ell \in \mathbb{N}$). Combining this with the above, we have, for $3 \mid n$,

$$\operatorname{sgn}\left(\sum_{d|n} \left(\frac{-1}{d}\right) d^2 + 6 \sum_{d|\frac{n}{3}} \left(\frac{-1}{d}\right) d^2\right) = (-1)^{\#\{p|n\,:\,p\equiv 3 \pmod 4,\, 2\nmid \operatorname{ord}_p(n)\}}.$$

Since $n \equiv 1 \pmod 4$, (5.1) again implies that the sign is 1. Substituting the first claim into the definition of $\mathcal{E}$ yields the second claim. □

5.3. **Finishing the proof of Theorem 1.3.** By Lemma 5.2, the signs of the Fourier coefficients of $\mathcal{E}$ agree with those asserted in Theorem 1.3. We are now ready to prove Theorem 1.3. Let $b_3(n) := C_{1^4 2^2 3^{-2}}(n)$ and $s_3(n) := \operatorname{sgn}(b_3(n))$.

Proof of Theorem 1.3. We claim that, for $n \in \mathbb{N}$, $s_3(n) = \operatorname{sgn}(A(4n+1))$. Theorem 5.2 then gives the claim. To show this, we explicitly compare the growth of the Fourier coefficients of $\mathcal{E}$ with the growth of the Fourier coefficients of the cuspidal part from Theorem 5.1. We begin by bounding the Fourier coefficients of the cuspidal part of $\frac{\eta(4z)^4\eta(8z)^4}{\eta(12z)^2}$. Write

$$g_1(z) =: \sum_{n\geq 1} a_1(n) q^n, \quad g_2(z) =: \sum_{n\geq 1} a_2(n) q^n.$$

Since g_1 and g_2 are weight 3 newforms, $|a_1(n)|$, $|a_2(n)| \leq d(n)n$ by Theorem 2.6. Moreover, by Theorem 5.1, for $n \equiv 1 \pmod 4$ the n-th Fourier coefficient of the cuspidal part of the

function on the right-hand side of Theorem 5.1 is

$$-\frac{12}{7}\delta_{3|n}a_1(n)+\delta_{3\nmid n}\frac{3}{14}\left(1+3\left(\frac{-3}{n}\right)\right)a_1(n)+\frac{3}{16}\left(-1+\left(\frac{-3}{n}\right)\right)a_2(n). \tag{5.3}$$

Note that $(\frac{-3}{n})=(\frac{n}{3})$. We now consider the residue classes of n (mod 12).

Assume first that $n\equiv 1\,(\mathrm{mod}\ 12)$. Then (5.3) equals $\frac{6}{7}a_1(n)$ and its absolute value can be bounded by $\frac{6}{7}d(n)n$. From Theorem 5.1, the absolute value of the n-th Fourier coefficient of the Eisenstein series part $\mathcal{E}$ is

$$\frac{1}{7}\left|\sum_{d|n}\left(\frac{-1}{d}\right)d^2\right|\geq\frac{1}{7}\prod_{p|n}\frac{p^{2\,\mathrm{ord}_p(n)+2}-1}{p^2+1}.$$

We conclude that $s_3(n)$ agrees with the claimed value if

$$d(n)\leq\frac{1}{6}\prod_{p|n}\frac{p^{2\,\mathrm{ord}_p(n)+2}-1}{p^{\mathrm{ord}_p(n)}\left(p^2+1\right)}.$$

Writing $n=\prod_{j=1}^{r}p_j^{\ell_j}$, we have $d(n)=\prod_{j=1}^{r}(\ell_j+1)$. Hence the above is equivalent to

$$\prod_{j=1}^{r}(\ell_j+1)\leq\frac{1}{6}\prod_{j=1}^{r}\frac{p_j^{2\ell_j+2}-1}{p_j^{\ell_j}\left(p_j^2+1\right)}. \tag{5.4}$$

We claim that, for a prime p and $\ell\in\mathbb{N}$,

$$\frac{p^{2\ell+2}-1}{p^\ell\left(p^2+1\right)}\geq\begin{cases}2.4(\ell+1) & \text{for } p=5 \text{ and } \ell=1,\\ 3.4(\ell+1) & \text{for } p=7 \text{ and } \ell=1,\\ 5(\ell+1) & \text{for } p=11 \text{ and } \ell=1,\\ 6(\ell+1) & \text{for } p\geq 13 \text{ or } (p\in\{5,7,11\} \text{ and } \ell\geq 2).\end{cases} \tag{5.5}$$

This clearly implies the claim unless n is one of the exceptional cases $n\in\{5,7,11\}$. But none of these satisfy $n\equiv 1\,(\mathrm{mod}\ 12)$. We now prove (5.5). We first directly verify (5.5) for $\ell=1$ and $p\in\{5,7,11\}$ by computing both sides. The claim for the remaining cases in (5.5) follows after showing that for $x\geq 13$ or $(x\in\{5,7,11\}$ and $\ell\geq 2)$

$$f_\ell(x):=x^{2\ell+2}-1-6(\ell+1)x^\ell\left(x^2+1\right)>0.$$

We do so by induction on ℓ for each x. First, for $x\geq 13$, we have

$$f_1(x)=x^4-1-12x\left(x^2+1\right)>0.$$

We also verify directly that $f_2(5)=3924>0$, $f_2(7)=73548>0$, and $f_2(11)=1505844>0$, so we see that the base case for the induction holds for each p. Next

$$f_{\ell+1}(x)=x^2f_\ell(x)+6(\ell+1)x^{\ell+1}\left(x^2+1\right)(x-1)+x^2-1-6x^{\ell+1}\left(x^2+1\right).$$

The first term is positive by induction. Using $x\geq 2$, we show that the remaining terms are non-negative. This proves (5.5) for $n\equiv 1\,(\mathrm{mod}\ 12)$.

Next assume that $n\equiv 5\,(\mathrm{mod}\ 12)$. Simplifying (5.3) in this case, the n-th Fourier coefficient of the cuspidal part is $-\frac{3}{7}a_1(n)-\frac{3}{8}a_2(n)$. Then Theorem 2.6 implies that the absolute value

is bounded by $\frac{45}{56}d(n)n$. With the same argument as above, we may conclude the claim if

$$\prod_{j=1}^{r}(\ell_j+1) \le \frac{56}{315}\prod_{j=1}^{r}\frac{p_j^{2\ell_j+2}-1}{p_j^{\ell_j}\left(p_j^2+1\right)}. \tag{5.6}$$

If (5.6) fails, then, since $\frac{56}{315} > \frac{1}{6}$, (5.4) does not hold. As shown for $n \equiv 1 \pmod{12}$, if (5.4) fails, then $n \in \{5, 7, 11\}$. Thus for $n > 5$ with $n \equiv 5 \pmod{12}$, we have $s_3(\frac{n-1}{4}) = \operatorname{sgn}(A(n))$, as claimed. For $n = 5$, a direct computer calculation shows that $s_3(1) = -1 = \operatorname{sgn}(A(5))$. Note that this is the only possible exceptional case satisfying $n \equiv 5 \pmod{12}$.

We finally consider $n \equiv 9 \pmod{12}$. Then, by (5.3), the Fourier coefficient of the cusp form is $-\frac{12}{7}a_1(n) - \frac{3}{16}a_2(n)$. By Theorem 2.6, the absolute value of this can be bounded by $\frac{213}{112}d(n)n$. Writing $n = 3^\ell m$ and comparing (5.2) with the Fourier coefficients of the Eisenstein series in the other congruence classes and simplifying, the Fourier coefficients of the Eisenstein series have the extra factor $\frac{1}{5}(3^{2\ell+1} + 7(-1)^{\ell+1})$. Bounding as above, $s_3(\frac{n-1}{4}) = \operatorname{sgn}(A(n))$ if

$$\frac{3^{2\ell+1}+7(-1)^\ell}{35}\prod_{p\mid m}\frac{p^{2\operatorname{ord}_p(n)+2}-1}{p^2+1} \ge \frac{213}{112}d(n)n.$$

Since $n = 3^\ell m$ and $d(n) = (\ell+1)d(m)$, this is equivalent to

$$\prod_{p\mid m}\frac{p^{2\operatorname{ord}_p(n)+2}-1}{p^2+1} \ge \frac{213}{112}\frac{35(\ell+1)3^\ell}{3^{2\ell+1}+7(-1)^\ell}d(m)m.$$

Note that

$$\frac{(\ell+1)3^\ell}{3^{2\ell+1}+7(-1)^\ell} \le \begin{cases} \frac{3}{10} & \text{if } \ell = 1, \\ \frac{27}{250} & \text{if } \ell = 2, \\ \frac{27}{545} & \text{if } \ell = 3, \\ \frac{81}{3938} & \text{if } \ell = 4, \\ \frac{729}{88750} & \text{if } \ell \ge 5. \end{cases}$$

We then extend (5.5) with

$$\frac{p^{2\ell+2}-1}{p^\ell\left(p^2+1\right)} \ge \begin{cases} 6.4(\ell+1) & \text{for } p = 13 \text{ and } \ell = 1, \\ 8(\ell+1) & \text{for } p = 5 \text{ and } \ell = 2, \\ 8.4(\ell+1) & \text{for } p = 17 \text{ and } \ell = 1, \\ 9.4(\ell+1) & \text{for } p = 19 \text{ and } \ell = 1, \\ 11.4(\ell+1) & \text{for } p = 23 \text{ and } \ell = 1, \\ 14.4(\ell+1) & \text{for } p = 29 \text{ and } \ell = 1, \\ 15.4(\ell+1) & \text{for } p = 31 \text{ and } \ell = 1, \\ 16(\ell+1) & \text{for } p = 7 \text{ and } \ell = 2, \\ 18.4(\ell+1) & \text{for } p = 37 \text{ and } \ell = 1, \\ 20(\ell+1) & \text{for } p \ge 41 \text{ or } (11 \le p \le 37 \text{ and } \ell \ge 2) \\ & \qquad \text{or } (p \in \{5, 7\} \text{ and } \ell \ge 3). \end{cases} \tag{5.7}$$

Combining (5.7) with (5.5), we then conclude that for $n \equiv 9 \pmod{12}$ with

$$n \notin \{9, 21, 33, 45, 57, 69, 81, 93, 105, 117, 165\},$$

we have $s_3(\frac{n-1}{4}) = \operatorname{sgn}(A(n))$. A computer calculation for $n \leq 165$ establishes the remaining cases. □

6. Proof of Theorem 1.4

6.1. **Proof of Theorem 1.4 (1).** Let $b_4(n) := C_{1^4 2^2 4^{-2}}(n)$ and $s_4 := \operatorname{sgn}(b_4(n))$. We first obtain a formula for the generating function of $b_4(n)$.

Lemma 6.1. *We have*

$$\frac{\eta(z)^4\eta(2z)^2}{\eta(4z)^2} = 1 - 4\sum_{n\geq 1}\left(\frac{-4}{n}\right)\sigma(n)q^n + 8\sum_{n\geq 1}(-1)^n\sigma(n)q^{4n} - 32\sum_{n\geq 1}\sigma(n)q^{16n}.$$

Proof. By [23, Theorem 1.64], $\frac{\eta(z)^4\eta(2z)^2}{\eta(4z)^2} \in M_2(\Gamma_0(16))$. The right-hand side of Lemma 6.1 is

$$-2E_{2,\chi_{-4},\chi_{-4}} + \frac{1}{3}E_2\big|V_4 - \frac{2}{3}E_2\big|U_2 \circ V_8 + \frac{4}{3}E_2\big|V_{16}. \tag{6.1}$$

One checks that (6.1) coincides with the corresponding identity with E_2 replaced by $\widehat{E}_2$. Thus, (6.1) is modular. By Lemmas 2.4 and 2.3 (1), (4), (6.1) lies in $M_2(\Gamma_0(16))$. Thus both sides are elements of $M_2(\Gamma_0(16))$. By Lemma 2.1, it suffices to verify it for the first 4 Fourier coefficients, which has been verified computationally. □

Proof of Theorem 1.4 (1). For $n = 0$, Lemma 6.1 gives $b_4(0) = 1 > 0$. We next let $n \in \mathbb{N}$ and split into cases depending on $\operatorname{ord}_2(n)$. By Lemma 6.1, if n is odd, then

$$b_4(n) = -4\left(\frac{-1}{n}\right)\sigma(n).$$

Since $4\sigma(n) > 0$, this gives

$$s_4(n) = -\left(\frac{-1}{n}\right) = \begin{cases} 1 & \text{if } n \equiv 3 \pmod 4, \\ -1 & \text{if } n \equiv 1 \pmod 4. \end{cases}$$

This gives the claim for n odd.

For $n \equiv 2 \pmod 4$, we see directly that none of the sums in Theorem 6.1 contributes to the n-th Fourier coefficient. This gives the claim in this case.

Next suppose that $4 \mid n$ but $16 \nmid n$. In this case, only the second sum in Theorem 6.1 contributes to the n-th Fourier coefficient, so we obtain

$$b_4(n) = -8\sigma\left(\frac{n}{4}\right).$$

This implies that $s_4(n) = -1$. This gives the corresponding case.

Finally, if $16 \mid n$, then the final two sums in Theorem 6.1 contribute to the n-th Fourier-coefficient, and we have

$$b_4(n) = 8\left(\sigma\left(\frac{n}{4}\right) - 4\sigma\left(\frac{n}{16}\right)\right).$$

We next write $n = 2^\ell m$ with $\ell \in \mathbb{N}_{\geq 3}$ and m odd and use multiplicativity of $\sigma(n)$ to simplify

$$\sigma\left(\frac{n}{4}\right) - 4\sigma\left(\frac{n}{16}\right) = 3\sigma(m).$$

Thus, for $16 \mid n$, we obtain

$$b_4(n) = 24\sigma(m) > 0.$$

Hence $s_4(n) = 1$ in this case, completing the proof. □

6.2. **Proof of Theorem 1.4 (2).** For $\boldsymbol{a} \in \mathbb{N}^\ell$, set

$$r_{\boldsymbol{a}}(n) := \#\Big\{\boldsymbol{n} \in \mathbb{Z}^\ell : \sum_{j=1}^{\ell} a_j n_j^2 = n\Big\}.$$

Using [23, Theorem 1.62, Propositions 1.41 and 3.7 (1)], Lemmas 2.2 and 2.1, we obtain the following identity.

Lemma 6.2. *We have*

$$\begin{aligned}\frac{\eta(z)^4\eta(2z)^4}{\eta(4z)^3} = &- \sum_{n\equiv 1 \pmod 4} r_{(1,1,2,2,2)}(n)q^n + \sum_{n\equiv 3 \pmod 4} r_{(1,1,2,2,2)}(n)q^n \\ &- \frac{1}{5}\sum_{n\equiv 2 \pmod{16}} r_{(1,1,2,2,2)}(n)q^n - \frac{1}{5}\sum_{n\equiv 4 \pmod{16}} r_{(1,1,2,2,2)}(n)q^n + \frac{1}{5}\sum_{n\equiv 6 \pmod{16}} r_{(1,1,2,2,2)}(n)q^n \\ &- \frac{3}{7}\sum_{n\equiv 10 \pmod{16}} r_{(1,1,2,2,2)}(n)q^n - \frac{1}{5}\sum_{n\equiv 12 \pmod{16}} r_{(1,1,2,2,2)}(n)q^n + \frac{1}{5}\sum_{n\equiv 14 \pmod{16}} r_{(1,1,2,2,2)}(n)q^n \\ &+ \frac{1}{5}\sum_{n\equiv 0 \pmod 8} \left(r_{(1,1,2,2,2)}(n) + 4r_{(1,1,2,2,2)}\left(\frac{n}{4}\right)\right) q^n.\end{aligned}$$

Proof of Theorem 1.4 (2). Liouville [21] proved that $r_{(1,1,2,2)}(n) > 0$ for every $n \in \mathbb{N}$. Since $r_{(1,1,2,2,2)}(n) \geq r_{(1,1,2,2)}(n)$, obtained by setting the fifth variable to zero, we have $r_{(1,1,2,2,2)}(n) > 0$ for all $n \in \mathbb{N}$. Theorem 1.4 (2) now follows immediately from Lemma 6.2. □

7. Proof of Theorem 1.5, Corollary 1.6, and Theorem 1.7

7.1. **Proof of Theorem 1.5.** We next bound and determine the sign of $\sum_{d|n}(\frac{d}{p})d^k$.

Lemma 7.1. *For $n = p^a m$ with $p \nmid m$ and $k \in \mathbb{N}$, we have*

$$\left|\sum_{d|n}\left(\frac{d}{p}\right)d^k\right| \gg_\varepsilon m^{k-\varepsilon}, \quad \operatorname{sgn}\left(\sum_{d|n}\left(\frac{d}{p}\right)d^k\right) = \left(\frac{m}{p}\right).$$

Proof. By multiplicativity, we have

$$\sum_{d|n}\left(\frac{d}{p}\right)d^k = \prod_{\ell|n}\sum_{j=0}^{\operatorname{ord}_\ell(n)}\left(\frac{\ell}{p}\right)^j \ell^{kj}.$$

Since $k \geq 1$, for $\ell \neq p$ and $r \in \mathbb{N}$ we have

$$\left|\sum_{j=0}^{r}\left(\frac{\ell}{p}\right)^j \ell^{kj}\right| \geq \ell^{kr} - \ell^{k(r-1)}.$$

Thus, for any $\varepsilon > 0$, we have, using [24, (3.27)] in the final step,

$$\left|\sum_{d|n}\left(\frac{d}{p}\right)d^k\right| \geq m^k \prod_{\ell|m}\left(1-\ell^{-k}\right) \gg_\varepsilon m^{k-\varepsilon}.$$

This proves the first claim. For the second claim, we compute

$$\sum_{j=0}^{r}\left(\frac{\ell}{p}\right)^j \ell^{kj} = \begin{cases} \frac{\ell^{(r+1)k}-1}{\ell^k-1} & \text{if } \left(\frac{\ell}{p}\right)=1, \\ \frac{\ell^{(r+1)k}+1}{\ell^k+1} & \text{if } \left(\frac{\ell}{p}\right)=-1,\ 2 \mid r, \\ -\frac{\ell^{(r+1)k}-1}{\ell^k+1} & \text{if } \left(\frac{\ell}{p}\right)=-1,\ 2 \nmid r, \\ 1 & \text{if } \ell = p. \end{cases}$$

Since $k \geq 1$, this is positive unless $(\frac{\ell}{p}) = -1$ and r is odd, in which case it is negative. Thus

$$\operatorname{sgn}\left(\sum_{d|n}\left(\frac{d}{p}\right)d^k\right) = (-1)^{\#\left\{\ell \text{ prime}:\left(\frac{\ell}{p}\right)=-1,\, 2\nmid \operatorname{ord}_\ell(m)\right\}}.$$

Now write $n = p^a m$ with $a \in \mathbb{N}_0$ and $\gcd(p, m) = 1$. The claim follows since

$$\left(\frac{m}{p}\right) = \prod_{\ell^r \| m}\left(\frac{\ell^r}{p}\right) = (-1)^{\#\left\{\ell \text{ prime}:\left(\frac{\ell}{p}\right)=-1,\, 2\nmid \operatorname{ord}_\ell(m)\right\}}. \qquad \square$$

Proof of Theorem 1.5. By [23, Theorem 1.64], we have $Q_p \in M_{\frac{p^2-p}{2}}(\Gamma_0(p), \chi_{(\frac{-1}{p})p})$ and $P_p \in M_{\frac{p-1}{2}}(\Gamma_0(p), \chi_{(\frac{-1}{p})p})$, so Theorem 2.5 yields that there exist $f_p \in S_{pk_p}(\Gamma_0(p), \chi_{(\frac{-1}{p})p})$ and $g_p \in S_{k_p}\ (\Gamma_0(p), \chi_{(\frac{-1}{p})p})$ such that

$$Q_p = L_{pk_p,p}E_{pk_p,1,\chi_{\left(\frac{-1}{p}\right)_p}} + f_p, \qquad P_p = L_{k_p,p}E_{k_p,1,\chi_{\left(\frac{-1}{p}\right)_p}} + g_p. \tag{7.1}$$

We split the Fourier expansions naturally into the two pieces

$$Q_p(z) =: \sum_{n\geq 0} c_{Q_p}(n)q^n =: \sum_{n\geq 0} A_{\mathrm{E}}(n)q^n + \sum_{n\geq 1} c_{f_p}(n)q^n, \tag{7.2}$$

$$P_p(z) =: \sum_{n\geq 0} c_{P_p}(n)q^n =: \sum_{n\geq 0} B_{\mathrm{E}}(n)q^n + \sum_{n\geq 1} c_{g_p}(n)q^n \tag{7.3}$$

corresponding to the right-hand sides of (7.1).
(1) Employing Theorem 2.7 and noting that f_p is uniquely determined by p, we have, recalling that $n = p^a m$ with $p \nmid m$,

$$|c_{f_p}(n)| \ll_{p,\varepsilon} \|f_p\|\, n^{\frac{pk_p-1}{2}+\varepsilon} \ll_p n^{\frac{pk_p-1}{2}+\varepsilon} \ll_{a,p,\varepsilon} m^{\frac{pk_p-1}{2}+\varepsilon}.$$

Using the bound in Theorem 7.1, we conclude from (7.2) that, for m sufficiently large,

$$\operatorname{sgn}\left(c_{Q_p}(n)\right) = \operatorname{sgn}\left(A_{\mathrm{E}}(n)\right). \tag{7.4}$$

By the evaluation of the sign in Theorem 7.1, we then obtain

$$\operatorname{sgn}\left(A_{\mathrm{E}}(n)\right) = \operatorname{sgn}\left(L_{pk_p,p}\right)\left(\frac{m}{p}\right).$$

Finally, using Lemma 2.9, we conclude that

$$\operatorname{sgn}\left(A_{\mathrm{E}}(n)\right) = \left(\frac{2}{p}\right)\left(\frac{m}{p}\right). \tag{7.5}$$

This finishes the proof of part (1).

(2) By Theorem 2.7, we have $|c_{g_p}(n)| \ll_{a,p,\varepsilon} m^{\frac{k_p-1}{2}+\varepsilon}$. As in part (1), Theorem 7.1 and (7.3) hence imply that, for m sufficiently large, we have

$$\operatorname{sgn}\left(c_{P_p}(n)\right) = \operatorname{sgn}\left(B_{\mathrm{E}}(n)\right) = \operatorname{sgn}\left(L_{k_p,p}\right)\left(\frac{m}{p}\right). \tag{7.6}$$

Finally, Lemma 2.9 implies that

$$\operatorname{sgn}\left(B_{\mathrm{E}}(n)\right) = \left(\frac{-2}{p}\right)\left(\frac{m}{p}\right), \tag{7.7}$$

finishing the proof. □

7.2. **Proof of Corollary 1.6.** Since (see [19])

$$S_3(\Gamma_0(3), \chi_{-3}) = \{0\}, \tag{7.8}$$

(7.1) and Theorem 2.8 yield the following identity for $b_5(n) := C_{1^9 3^{-3}}(n)$.

Lemma 7.2. *We have*

$$\frac{\eta(z)^9}{\eta(3z)^3} = 1 - 9\sum_{n\geq 1}\sum_{d\mid n}\left(\frac{d}{3}\right)d^2 q^n.$$

In particular, for $n \in \mathbb{N}$ we have

$$b_5(n) = -9\sum_{d\mid n}\left(\frac{d}{3}\right)d^2.$$

We next prove Corollary 1.6 (1). Let $s_5(n) := \operatorname{sgn}(b_5(n))$ and $s_6(n) := \operatorname{sgn}(C_{1^5 5^{-1}}(n))$.

Proof of Corollary 1.6. (1) Since $f_3 = 0$ in (7.1) by (7.8), (7.2) implies that (7.4) holds for $n \in \mathbb{N}$, and hence (7.5) gives

$$s_5(n) = \operatorname{sgn}\left(c_{Q_3}(n)\right) = \operatorname{sgn}\left(A_{\mathrm{E}}(n)\right) = \left(\frac{2}{3}\right)\left(\frac{m}{3}\right) = -\left(\frac{m}{3}\right).$$

(2) Since $S_2(\Gamma_0(5), \chi_5) = \{0\}$ (see [19]), we similarly conclude from (7.3) that (7.6) holds for $n \in \mathbb{N}$ and thus (7.7) yields that $s_6(n) = -(\frac{m}{5})$. □

7.3. **Proof of Theorem 1.7.** The first step in proving Theorem 1.7 is to write our functions in terms of class number generating functions. We have the following.

Lemma 7.3. *We have*

$$\frac{\eta(8z)^2\eta(16z)^2}{\eta(24z)} = (\mathcal{H}_{4,3} - \mathcal{H}_{1,3}) \mid S_{24,1}(z) - \frac{1}{2}(\mathcal{H}_{4,3} - \mathcal{H}_{1,3}) \mid S_{24,17}(z) - (\mathcal{H}_{4,3} + 2\mathcal{H}_{1,3}) \mid S_{24,9}(z).$$

Proof. By Theorem 2.10, $\mathcal{H}_{1,3} \in M_{\frac{3}{2}}(\Gamma_0(12), \chi_{12})$ and $\mathcal{H}_{4,3} \in M_{\frac{3}{2}}(\Gamma_0(24), \chi_{12})$. By Theorem 2.2 (2), the right-hand side of the lemma lies in $M_{\frac{3}{2}}(\Gamma_0(576), \chi_{12})$. Moreover, since $\eta(24z) \in M_{\frac{1}{2}}(\Gamma_0(576), \chi_{12})$ (see [23, Corollary 1.62]), $\eta(24z)$ times the right-hand side is in $M_2(\Gamma_0(576))$. Next, by [23, Theorem 1.64], $\eta(8z)^2\eta(16z)^2 \in M_2(\Gamma_0(64))$. Thus $\eta(24z)$ times

the difference of the left- and right-hand sides lies in $M_2(\Gamma_0(576))$. By Theorem 2.1, we have to check 192 Fourier coefficients. The claim was verified computationally by checking the first 192 Fourier coefficients. □

Proof of Theorem 1.7. Let $b_7(n) := C_{1^2 2^2 3^{-1}}(n)$ and $s_7(n) := \operatorname{sgn}(b_7(n))$. Let

$$\frac{\eta(8z)^2\eta(16z)^2}{\eta(24z)} =: \sum_{n\geq 0} C(n)q^n.$$

Then $b_7(n) = C(8n+1)$. Thus

$$s_7(n) = \operatorname{sgn}(C(8n+1)).$$

By (2.4), we have

$$\mathcal{H}_{\ell_1,\ell_2}(z) = \sum_{n\geq 0}\left(H(\ell_1\ell_2 n) - \ell_2 H\left(\frac{\ell_1 n}{\ell_2}\right)\right)q^n.$$

Hence Theorem 7.3 implies that, for $n = 3\ell$, we have

$$b_7(3\ell) = H(12(24\ell+1)) - H(3(24\ell+1)).$$

We write $3(24\ell+1) = Df^2$ and $12(24\ell+1) = D(2f)^2$ with $-D$ a fundamental discriminant and $f \in \mathbb{N}$. Then (2.2) implies that

$$H(3(24\ell+1)) = H(D)S_D(f), \quad H(12(24\ell+1)) = H(D)S_D(2f).$$

Since $S_D(f)$ is multiplicative and f is odd, we have

$$S_D(2f) - S_D(f) = S_D(f)S_D(2) - S_D(f) = S_D(f)\left(\sigma(2) - \left(\frac{-D}{2}\right) - 1\right) > 0.$$

So $s_7(3\ell) = 1 = (\frac{8\cdot n+1}{3})$, and we are done in this case. Similarly $b_7(3\ell+2) = C(24\ell+17)$. The same argument gives $s_7(3\ell+2) = -1 = (\frac{8n+1}{3})$ for $n = 3\ell+2$. This finishes the case $a = 0$, as $3 \mid (8n+1)$ iff $n \equiv 1 \pmod 3$, where we write $8n+1 = 3^a m$ with $3 \nmid m$ as in the theorem.

We hence next assume that $n = 3\ell+1$. In this case, Theorem 7.3 implies that

$$b_7(3\ell+1) = C(24\ell+9) = -H(36(8\ell+3)) + 3H\left(4(8\ell+3)\right) - 2H(9(8\ell+3)) + 6H(8\ell+3).$$

Write $3^{a-1}m = 8\ell+3 = Df^2$ with $-D$ fundamental and $f \in \mathbb{N}$. Using (2.2) gives

$$C(24\ell+9) = -H(D)(S_D(6f) - 3S_D(2f) + 2S_D(3f) - 6S_D(f)). \tag{7.9}$$

Note that $2 \nmid f$. Since D is a fundamental discriminant, we have $9 \nmid D$, so $a \geq 3$ if and only if $3 \mid f$. Moreover, $9 \mid (8\ell+3)$ if and only if $3 \mid f$. Therefore

$$a \geq 3 \Leftrightarrow 3 \mid f \Leftrightarrow 8\ell+3 \equiv 0 \pmod 9 \Leftrightarrow \ell \equiv 3 \pmod 9.$$

We next assume that $a \in \{1,2\}$, and hence $\ell \not\equiv 3 \pmod 9$. Since $S_D(f)$ is multiplicative, using (2.3) and (2.2), we have, by (7.9),

$$\begin{aligned} C(24\ell+9) &= -H(D)S_D(f)(S_D(6) - 3S_D(2) + 2S_D(3) - 6) \\ &= -H(8\ell+3)\bigg(\sigma(6) - \left(\frac{-D}{3}\right)\sigma(2) - \left(\frac{-D}{2}\right)\sigma(3) + \left(\frac{-D}{6}\right) - 3\sigma(2) \\ &\qquad + 3\left(\frac{-D}{2}\right) + 2\sigma(3) - 2\left(\frac{-D}{3}\right) - 6\bigg). \end{aligned}$$

Note that $(\frac{-D}{2}) = (\frac{-3}{2}) = -1$ since $D \equiv 3 \pmod 8$ and $(\frac{-D}{3}) = (\frac{-8\ell}{3}) = (\frac{n}{3})$. Thus

$$C(24\ell + 9) = -6H(8\ell + 3)\left(1 - \left(\frac{\ell}{3}\right)\right).$$

This proves the claim for $a \in \{1, 2\}$.

Finally, suppose that $a \geq 3$, which is equivalent to $n \equiv 3 \pmod 9$. We write $f = 3^r g$ with $3 \nmid g$. By (7.9) and the multiplicativity of $S_D(f)$ (splitting off the 3-powers), we have

$$C(24\ell + 9) = -H(D)S_D(g)\left(S_D\left(2 \cdot 3^{r+1}\right) - 3S_D\left(2 \cdot 3^r\right) + 2S_D\left(3^{r+1}\right) - 6S_D\left(3^r\right)\right).$$

We compute

$$S_D\left(3^\nu\right) = \sigma\left(3^\nu\right) - \left(\frac{-D}{3}\right)\sigma\left(3^{\nu-1}\right), \quad S_D(2) = \sigma(2) - \left(\frac{-D}{2}\right) = 3 - (-1) = 4.$$

Simplifying gives

$$-H(D)S_D(g)\left(6S_D\left(3^{r+1}\right) - 18S_D\left(3^r\right)\right) = -6H(D)S_D(g)\left(1 - \left(\frac{-D}{3}\right)\right).$$

Noting that the right-hand side is independent of r, the result follows. □

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